Reading Time: An initial **2 minutes** to view **BOTH** sections

	MATHEMATICS METHODS : UNITS 3 & 4, 2023 **Test 2 – Integrals, FTC and Exponential Functions (10%)** **3.2.4, 3.2.6 – 3.2.22, 3.1.1 – 3.1.4, 3.1.9**(exponentials only)	PM
Time Allowed 25 minutes	**First Name** **Surname**	**Marks** 26 marks

Circle your Teacher's Name: Mrs Alvaro Ms Chua Mrs Fraser-Jones

Mrs Greenaway Mr Luzuk Mrs Murray

Ms Narendranathan Mr Tanday

Assessment Conditions: ***(N.B. Sufficient working out must be shown to gain full marks)***

- Calculators: Not Allowed
- Formula Sheet: Provided
- Notes: Not Allowed

PART A – CALCULATOR FREE

QUESTION 1 **(1, 1, 3: 5 marks)**

Differentiate the following, giving your answer in simplest form:

a) $y = 2e^{x^2+x}$

$y' = 2(2x+1)e^{x^2+x}$
or
$y' = (4x+4)e^{x^2+x}$

✓ correctly differentiates

b) $\int_{-2}^{x} \frac{e^t}{t^3}\, dt$

$$\frac{d}{dx}\left(\int_{-2}^{x} \frac{e^t}{t^3}\, dt\right) = \boxed{\frac{e^x}{x^3}}$$

✓ applies FTC to obtain correct answer

c) $f(x) = \frac{e^{3x^4}}{2x^2}$

$$f'(x) = \frac{(2x^2)(12x^3e^{3x^4}) - (e^{3x^4})(4x)}{(2x^2)^2}$$

$$= \frac{24x^5e^{3x^4} - 4xe^{3x^4}}{4x^4}$$

$$= \frac{4xe^{3x^4}(6x^4 - 1)}{4x^4}$$

$$= \boxed{\frac{e^{3x^4}(6x^4 - 1)}{x^3}}$$

✓ attempts to use quotient rule
✓ correctly uses quotient rule
✓ fully simplifies

QUESTION 2 -1 overall if no 'c' in Q2 a & c **(2, 2, 3 - 7 marks)**

Determine the following:

a) $\int 12x^3 e^{x^4+5}\,dx$

$= 3\int 4x^3 e^{x^4+5}\,dx$ ✓ 3

$= \boxed{3e^{x^4+5} + c}$ ✓ correctly integrates exponential

$\int_1^0 \frac{d}{dx}[4\pi x \cdot e^{3x}]dx$

$-\int_0^1 \frac{d}{dx}[4\pi x \cdot e^{3x}]dx = \boxed{-4\pi e^3}$

✓ Reverses bounds and makes negative
✓ correctly answer

or

$= [4\pi x \cdot e^{3x}]_1^0$

$= 0 - 4\pi e^3$

$= \boxed{-4\pi e^3}$

✓ Substitutes correctly
✓ correctly answer

b) $\int 3x(x^2-1)^4 dx$

$= \frac{3}{10}\int 10x(x^2-1)^4 dx$

$= \boxed{\frac{3(x^2-1)^5}{10} + c}$

✓ valid method to find $10x(x^2-1)^4$
✓ correct multiplier
✓ correctly integrates

$$\frac{d}{dx}(x^2-1)^5 = 5\times 2x(x^2-1)^4$$
$$= 10x(x^2-1)^4$$

Alt. Sol

$= \frac{3}{2}\int 2x(x^2-1)^4 dx$ ✓

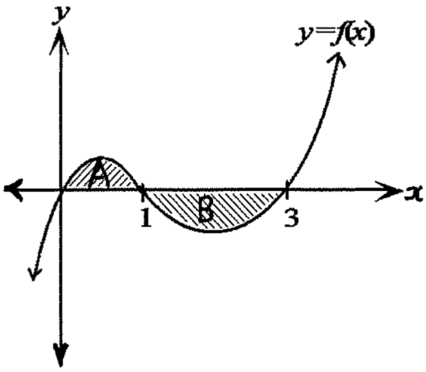
$= \frac{3}{10}\times\frac{(x^2-1)^5}{5}$ ✓

$= \frac{3(x^2-1)^5}{10} + c$ ✓

QUESTION 3 **(1, 1, 2, 2 – 6 marks)**

The graph of $y = f(x)$ is shown below. The area of the shaded region *A* is 12 square units and of region *B* is 24 square units.

Evaluate the following.



a) $\int_0^1 2\,f(x)\,dx$

$= 2\int_0^1 f(x)\,dx$

$= \boxed{24}$ ✓

b) $\int_0^3 f(x)\,dx = 12 - 24$

$= \boxed{-12}$ ✓

c) $\int_0^1 [\,1 + f(x)]\,dx = \int_0^1 1\,dx + \int_0^1 f(x)dx$

$= 1 + 12$

$= \boxed{13}$

✓ expresses integral as sum of $\int_0^1 1\,dx$ and $\int_0^1 f(x)dx$ - must be shown for full marks
✓ evaluates integral correctly

d) $\int_1^3 f'(x)\,dx = f(3) - f(1)$

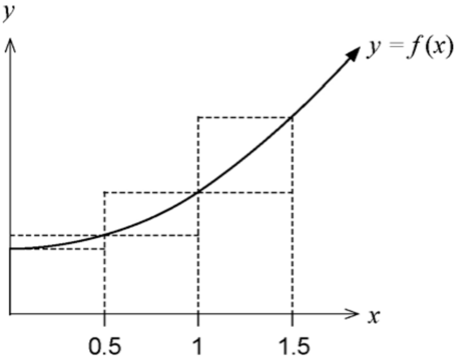
$= 0 - 0$

$= \boxed{0}$

✓ applies fundamental theorem of calculus - must be shown for full marks
✓ evaluates correctly

QUESTION 4 **(3 marks)**

Consider the function $f(x)$ shown graphed below. The table gives the value of the function at the given x values.



x	0	0.5	1	1.5
$f(x)$	18	19	22	27

By considering the areas of the rectangles shown, demonstrate and explain why $29.5 < \int_0^{1.5} f(x)dx < 34$.

Lower limit $= 18 \times 0.5 + 19 \times 0.5 + 22 \times 0.5 = 9 + 9.5 + 11 = 29.5$

Upper limit $= 19 \times 0.5 + 22 \times 0.5 + 27 \times 0.5 = 9.5 + 11 + 13.5 = 34$

OR

Inscribed (underestimate) rectangles $= 0.5(18 + 19 + 22) = \frac{59}{2} = 29.5$

Circumscribed (overestimate) rectangles $= 0.5(19 + 22 + 27 = \frac{68}{2} = 34$

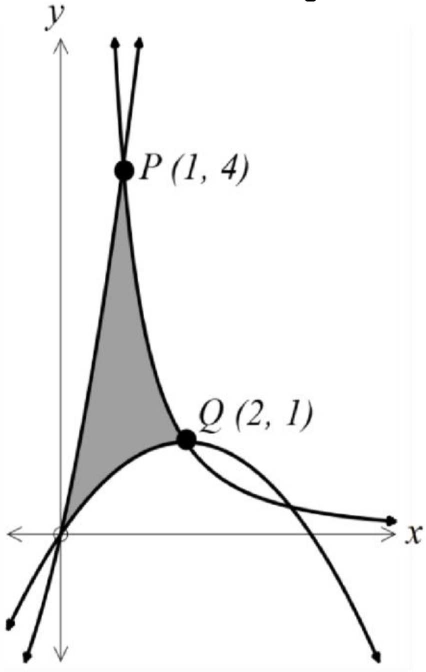
✓ shows a calculation to produce an underestimate of area
✓ shows a calculation to produce an overestimate of area

$\therefore \int_0^{1.5} f(x)dx$ is the area enclosed by curve and x-axis between $x = 0$ and $x = 1.5$, thus must lie between the rectangle areas. Hence, $29.5 < \int_0^{1.5} f(x)dx < 34$, as required.

✓ explains the limits in terms of area

QUESTION 5 **(5 marks)**

The origin, O, and the points P and Q are the vertices of the curved 'triangle' which is shaded in the diagram. The sides lie on curves with equations $y = x(x+3)$, $y = x - \frac{x^2}{4}$ and $y = \frac{4}{x^2}$. Calculate the area of the shaded region.



$$A = \int_0^1 x(x+3) - \left(x - \frac{x^2}{4}\right) dx + \int_1^2 \frac{4}{x^2} - \left(x - \frac{x^2}{4}\right) dx$$

$$= \int_0^1 \left(x^2 + 3x - x + \frac{x^2}{4}\right) dx + \int_1^2 \left(4x^{-2} - x + \frac{x^2}{4}\right) dx$$

$$= \int_0^1 \left(\frac{5x^2}{4} + 2x\right) dx + \left[-4x^{-1} - \frac{x^2}{2} + \frac{x^3}{12}\right]_1^2$$

$$= \left[\frac{5x^3}{12} + x^2\right]_0^1 + \left(-2 - 2 + \frac{8}{12}\right) - \left(-4 - \frac{1}{2} + \frac{1}{12}\right)$$

$$= \frac{5}{12} + 1 - 4 + \frac{8}{12} + 4 + \frac{1}{2} - \frac{1}{12}$$

$$= 2\frac{1}{2}\ units^2$$

✓ correctly sets up first integral
✓ correctly sets up second integral
✓ correctly integrates and substitutes into first integral
✓ correctly integrates and substitutes into integral
✓ correct area

Allow F/T

Units question

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MATHEMATICS METHODS : UNITS 3 & 4, 2023

Test 2 – Integrals, FTC and Exponential Functions (10%)

3.2.4, 3.2.6 – 3.2.22, 3.1.1 – 3.1.4, 3.1.9(exponentials only)

PM

Time Allowed 20 minutes	First Name	Surname	Marks 19 marks

Circle your Teacher's Name: Mrs Alvaro, Ms Chua, Mrs Fraser-Jones, Mrs Greenaway, Mr Luzuk, Mrs Murray, Ms Narendranathan, Mr Tanday

Assessment Conditions: ***(N.B. Sufficient working out must be shown to gain full marks)***

- Calculators: Not Allowed
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PART B – CALCULATOR ASSUMED

QUESTION 6 **(3, 2, 2: 7 marks)**

In January 1995 the purebred dingo population on Fraser Island was 300. The population, P, since then can be modelled by;

$$P = 80 + Ae^{kt}$$

where A and k are constants, and t is the number of years since January 1995.

a) Show that $\frac{dP}{dt} = k(P - 80)$

$P = 80 + Ae^{kt} \Rightarrow Ae^{kt} = P - 80$

$\frac{dP}{dt} = Ake^{kt}$

$= k(Ae^{kt})$

$= k(P - 80)$

✓ Rearranges $P = 80 + Ae^{kt}$
✓ Differentiates P with respect to t
✓ Substitutes $P - 80$ into equation

Alt. Sol

$\frac{dP}{dt} = Ake^{kt} = k \times Ae^{kt}$ ✓

$= k(80 + Ae^{kt} - 80)$ ✓

$= k(P - 80)$ ✓

b) In January 2015 it was found that the purebred population had dropped to 162. Show that the purebred dingo population is decreasing at an annual rate of approximately 5% per year.

When $t = 0$

$300 = 80 + Ae^0$

$= 80 + A$

$A = 220$

$t = 20, P = 162$

$162 = 80 + 220e^{20k}$

$k = -0.0493$

$k \approx -0.05$

✓ Finds A
✓ Substitutes into correct equation
✓ States approximate decrease

∴ the population is decreasing at a rate of approximately 5% per year.

c) Assuming this pattern continues, what will the purebred dingo population be in January 2050?

When $t = 55$

$P = 80 + 220e^{55k}$

$= 94.58$

≈ 95

✓ Correctly substitutes into equation
✓ States population: accept 94

∴ In 2050 the population is predicted to be 95 purebred dingoes.

QUESTION 7 **(2 marks)**

An oil storage tank ruptures at a time $t = 0$ and oil leaks from the tank at a rate of $r(t) = -100e^{-0.01t}$ litres per minute. How much oil leaks out during the third hour?

$$\int_{120}^{180} -100e^{-0.01t}\, dt = -1358.95\ L$$

✓ Correct limits of integration
✓ Correct answer

∴ 1358.95 L leak out in the third hour.

QUESTION 8 **(4 marks)**

A product is sold such that the price per unit is given by $p = -3x^2 + 600x$ dollars when x units are sold. Find the marginal revenue at $x = 300$ units and interpret the result.

revenue function: $R(x) = p\,.\,x = (-3x^2 + 600x)\,.\,x = -3x^3 + 600x^2$

marginal revenue: $R'(x) = \dfrac{dR}{dx} = -9x^2 + 1200x$

marginal revenue at $x = 300 \implies R'(300) = \left.\dfrac{dR}{dx}\right|_{x=300} = -9(300)^2 + 1200(300) = -450\,000$

Interpretation: If production increases from 300 to 301 units, the revenue decreases by 450 000 dollars.

This means that at the point when 300 units have been sold, sell the next unit will approximately result in decrease in revenue by $450 000.

✓ Finds revenue function
✓ Finds marginal revenue function
✓ Finds marginal revenue at x = 300
✓ Correctly interprets result

QUESTION 9 **(3 marks)**

A particle starts from rest and moves in a straight line with an acceleration of $a(t) = t^2 + e^{t-1} - 5\ m/s^2$. What is the total distance travelled in the 2nd second, to the nearest m?

$$v(t) = \frac{t^3}{3} + e^{t-1} - 5t - \frac{1}{e}$$

$$x(2) - x(1) = \int_1^2 \left|\frac{t^3}{3} + e^{t-1} - 5t - \frac{1}{e}\right| = 4.9\ m \approx 5m$$

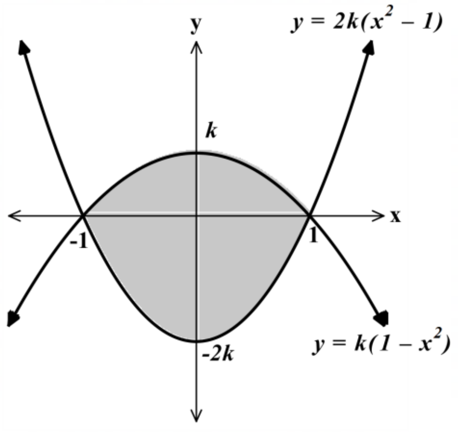
The distance travelled is 5 m during the 2nd second.

✓ finds $v(t)$
✓ correct bounds
✓ finds distance to nearest m

Rounding Question

QUESTION 10 **(3 marks)**

The shaded region shown is enclosed by two parabolas, each with x-intercepts at $x = -1$ and $x = 1$. Given that the area of the shaded region is 8, find the value of k, where $k > 0$.



$$A = 2\int_0^1 k(1 - x^2) - 2k(x^2 - 1)dx = 8$$

$\therefore k = 2$

✓ Sets up integral with bounds 0 and 1
✓ Recognises needs to multiply integral by 2
✓ correctly solves for k

Students cannot not use absolute value when trying to solve on ClassPad, max 1/3 if they try.